

华东交通大学考试专用答题纸 (适用于匿名改卷)

课程: 6.2 定积分换元法

题号	一	二	三	四	五	六	七	八	九	十	总分
得分											
阅卷人											

考生注意: 答题时应注明题号, 并按题号顺序答题。

题号	答 题 内 容
P114	廖国勇
1	$\int_{-2}^2 x^2 \sin^3 x dx = 0$ 奇函数在对称区间上的定积分为 0
2	$\int_{-1}^1 e^{ x } dx = 2 \int_0^1 e^x dx = 2e^x \Big _0^1 = 2e - 2$
3	设 $F(x) = \int_0^x \sin(x-t)^2 dt$, 则 $F'(x) =$ _____ 解 $F(x) =$ 设 $x-t = u$, 则 $t = -u + x$, $dt = -du$ 当 $t=0$ 时, $u=x$; 当 $t=x$ 时, $u=0$, 于是 $F(x) = \int_0^x \sin(x-t)^2 dt = \int_x^0 \sin u^2 (-du) = \int_0^x \sin u^2 du$ $F'(x) = \sin x^2$
4	定积分 $\int_{-1}^1 (x^2 + \sin^3 x) dx =$ _____ 解: 原式 $= \int_{-1}^1 x^2 dx + \int_{-1}^1 \sin^3 x dx$ $= 2 \int_0^1 x^2 dx + 0$ $= \frac{2}{3} x^3 \Big _0^1$ $= \frac{2}{3}$

题号	答题内容	廖国勇
二 1	求定积分 $\int_0^2 \sqrt{4-x^2} dx$ 解 设 $x=2\sin\theta$, 则 $dx=2\cos\theta d\theta$ 当 $x=0$ 时, $\theta=0$; 当 $x=2$ 时, $\theta=\frac{\pi}{2}$, 于是 $\begin{aligned} \int_0^2 \sqrt{4-x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2} \cdot 2\cos\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 4\cos^2\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} (2+2\cos 2\theta) d\theta \\ &= (2\theta + \sin 2\theta) \Big _0^{\frac{\pi}{2}} \\ &= \pi \end{aligned}$ 别法 $\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta + \sin^2\theta}{2} d\theta$ $= \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi}{4}$	
2	求定积分 $\int_1^{\sqrt{3}} \frac{1}{x^2 \sqrt{x^2+1}} dx$ 解: 设 $x=\tan\theta$, 则 $dx=\sec^2\theta d\theta$ 当 $x=1$ 时, $\theta=\frac{\pi}{4}$; $x=\sqrt{3}$ 时, $\theta=\frac{\pi}{3}$, 于是 $\begin{aligned} \int_1^{\sqrt{3}} \frac{1}{x^2 \sqrt{x^2+1}} dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta \cdot \sec\theta} \sec^2\theta d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta} d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^{-2}\theta d\sin\theta \\ &= \left(-\frac{1}{\sin\theta}\right) \Big _{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \left[-\left(\frac{\sqrt{3}}{2}\right)^{-1} - \left(-\right)\left(\frac{\sqrt{2}}{2}\right)^{-1}\right] \\ &= \sqrt{2} - \frac{2}{\sqrt{3}} \\ &= \sqrt{2} - \frac{2\sqrt{3}}{3} \end{aligned}$	

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二 3	求定积分 $\int_{\sqrt{2}}^2 \frac{1}{x^2 \sqrt{x^2-1}} dx$	
	解 设 $x = \sec \theta$, 则 $dx = \sec \theta \cdot \tan \theta d\theta$	
	当 $x = \sqrt{2}$ 时, $\theta = \frac{\pi}{4}$; 当 $x = 2$ 时 $\theta = \frac{\pi}{3}$	
	于是, 原式 $= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2 \theta \cdot \tan \theta} \sec \theta \cdot \tan \theta d\theta$	
	$= \int_{\pi/4}^{\pi/3} \cos \theta d\theta$	
	$= \sin \theta \Big _{\pi/4}^{\pi/3}$	
	$= \frac{\sqrt{3} - \sqrt{2}}{2}$	
4	求定积分 $\int_0^1 \frac{1}{1+\sqrt{1-x}} dx$	
	解: 设 $x = \sin^2 \theta$, 则 $dx = 2 \sin \theta \cos \theta d\theta$	
	当 $x=0$ 时, $\theta=0$; 当 $x=1$ 时, $\theta = \frac{\pi}{2}$, 于是	
	$\int_0^1 \frac{1}{1+\sqrt{1-x}} dx = \int_0^{\pi/2} \frac{1}{1+\cos \theta} 2 \sin \theta \cos \theta d\theta$	
	$= \int_0^{\pi/2} \frac{-2 \cos \theta}{1+\cos \theta} d\cos \theta$	
	$= \int_0^{\pi/2} \left[\frac{-2(1+\cos \theta)}{1+\cos \theta} + \frac{2}{1+\cos \theta} \right] d\cos \theta$	
	$= \int_0^{\pi/2} -2 d\cos \theta +$	
	$= \int_0^{\pi/2} -2 d\cos \theta + 2 \int_0^{\pi/2} \frac{1}{1+\cos \theta} d(1+\cos \theta)$	
	$= -2 \cos \theta \Big _0^{\pi/2} + 2 \ln(1+\cos \theta) \Big _0^{\pi/2}$	
	$= 2 + (2 \ln 2)$	
	$= 2 - 2 \ln 2$	
	别解: 设 $t = \sqrt{1-x}$, 则 $x = 1-t^2$, $dx = -2t dt$ $x=0$ 时 $t=1$; $x=1$ 时, $t=0$, 于是原式 $= \int_1^0 \frac{1}{1+t} (-2t dt) = \int_0^1 \frac{2t}{1+t} dt$ $= \int_0^1 2 dt - 2 \int_0^1 \frac{1}{1+t} dt = 2 - 2 \ln 2$	

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P115

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二 5

求定积分 $\int_0^{\ln 2} \frac{1}{\sqrt{e^x-1}} dx$ 解 设 $t = \sqrt{e^x-1}$, 则 $x = \ln(1+t^2)$, $dx = \frac{2t}{1+t^2} dt$ 当 $x \rightarrow 0$ 时, $t \rightarrow 0$; 当 $x = \ln 2$ 时 $t = 1$ 于是 $\int_0^{\ln 2} \frac{1}{\sqrt{e^x-1}} dx = \int_0^1 \frac{1}{t} \cdot \frac{2t}{1+t^2} dt$

$$= \int_0^1 2 \frac{1}{1+t^2} dt$$

$$= 2 \arctan t \Big|_0^1$$

$$= \frac{\pi}{2}$$

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6 $f(x) = \begin{cases} 2e^x, & x \leq 0 \\ 3x^2, & x > 0 \end{cases}$ 求 $\int_{-2}^3 f(x-1) dx$ 解: 设 $y = x-1$, 则 $x = y+1$, $dx = dy$ 当 $x = -2$ 时, $y = -3$; 当 $x = 3$ 时, $y = 2$ 于是 $\int_{-2}^3 f(x-1) dx = \int_{-3}^2 f(y) dy$

$$= \int_{-3}^0 f(y) dy + \int_0^2 f(y) dy$$

$$= \int_{-3}^0 2e^y dy + \int_0^2 3y^2 dy$$

$$= 2e^y \Big|_{-3}^0 + y^3 \Big|_0^2$$

$$= (2 - 2e^{-3}) + (8 - 0)$$

$$= 10 - \frac{2}{e^3}$$

7 求定积分 $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$ 解: 设 $x = \sin \theta$, 则 $dx = \cos \theta d\theta$, 当 $x = 0$ 时 $\theta = 0$, 当 $x = \frac{\sqrt{2}}{2}$ 时, $\theta = \frac{\pi}{4}$

$$\text{原式} = \int_0^{\frac{\pi}{4}} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right) \Big|_0^{\frac{\pi}{4}} = \left(\frac{\pi}{8} - \frac{1}{4} \right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

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= 8

$$\text{设 } f(x) = \begin{cases} x \sin^2 x & x < 1 \\ \frac{\sqrt{x^2-1}}{x} & x \geq 1 \end{cases} \quad \text{求 } \int_{-1}^{\sqrt{2}} f(x) dx$$

$$\begin{aligned} \text{解 } \int_{-1}^{\sqrt{2}} f(x) dx &= \int_{-1}^1 f(x) dx + \int_1^{\sqrt{2}} f(x) dx \\ &= \int_{-1}^1 x^2 \sin x dx + \int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x} dx \quad x^2 \sin x \text{ 为奇函数} \\ &= \int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x} dx \end{aligned}$$

$$\text{设 } x = \sec \theta, \text{ 则 } dx = \sec \theta \cdot \tan \theta d\theta \quad x=1 \text{ 时 } \theta = 0; x=\sqrt{2} \text{ 时 } \theta = \frac{\pi}{4}$$

$$\begin{aligned} \text{于是, 原式} &= \int_0^{\pi/4} \frac{\tan \theta}{\sec \theta} \cdot \sec \theta \tan \theta d\theta \\ &= \int_0^{\pi/4} \tan^2 \theta d\theta \\ &= \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta \\ &= (\tan \theta - \theta) \Big|_0^{\pi/4} \\ &= 1 - \frac{\pi}{4} \end{aligned}$$

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$$9 \text{ 设 } F(x) = \int_0^x t f(x^2 - t^2) dt, \text{ 求 } F'(x)$$

$$\text{解: 设 } y = x^2 - t^2, \text{ 则 } dy = -2t dt$$

$$\text{当 } t=0 \text{ 时, } y=x^2; \text{ 当 } t=x \text{ 时, } y=0$$

$$\begin{aligned} F(x) &= \int_0^x f(x^2 - t^2) \left(-\frac{1}{2}\right) (-2t dt) \\ &= \int_{x^2}^0 f(y) \left(-\frac{1}{2}\right) dy = \frac{1}{2} \int_0^{x^2} f(y) dy \end{aligned}$$

$$\begin{aligned} F'(x) &= \frac{1}{2} \cdot f(x^2) \cdot 2x \\ &= x f(x^2) \end{aligned}$$

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三 10 求极限 $\lim_{x \rightarrow 0} \frac{\int_0^x \sin(xt)^2 dt}{x^5}$ 解: 设 $y = xt$, 则 $t = \frac{y}{x}$, $dt = \frac{1}{x} dy$ 当 ~~$t=0$~~ $t=0$ 时, $y=0$; 当 $t=x$ 时, $y=x^2$ 于是 $\int_0^x \sin(xt)^2 dt = \int_0^{x^2} \sin y^2 \cdot \frac{1}{x} dy = \frac{1}{x} \int_0^{x^2} \sin y^2 dy$ 原式 = $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin y^2 dy}{x^6}$

$$= \lim_{x \rightarrow 0} \frac{(\sin x^4) \cdot 2x}{6x^5} = \lim_{x \rightarrow 0} \frac{x^4 \cdot 2x}{6x^5} = \frac{1}{3}$$

11 求定积分 $\int_0^{\frac{\pi}{4}} \ln(1+\tan x) dx$ 解: 原式 = $\int_0^{\frac{\pi}{4}} \ln \frac{\cos x + \sin x}{\cos x} dx$

$$= \int_0^{\frac{\pi}{4}} \ln \frac{\sqrt{2} \sin(x+\frac{\pi}{4})}{\cos x} dx$$

$$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2} dx + \int_0^{\frac{\pi}{4}} \sin(x+\frac{\pi}{4}) dx - \int_0^{\frac{\pi}{4}} \cos x dx$$

$$\int_0^{\frac{\pi}{4}} \sin(x+\frac{\pi}{4}) dx = \int_0^{\frac{\pi}{4}} \sin[(0+\frac{\pi}{4}-x)+\frac{\pi}{4}] dx$$

翻转公式

$$= \int_0^{\frac{\pi}{4}} \sin(\frac{\pi}{2}-x) dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x dx$$

$$\text{于是, 原式} = \int_0^{\frac{\pi}{4}} \ln \sqrt{2} + \int_0^{\frac{\pi}{4}} \cos x dx - \int_0^{\frac{\pi}{4}} \cos x dx$$

$$= \frac{\pi}{4} \ln \sqrt{2}$$

$$= \frac{\ln 2}{8} \pi$$

注 翻转公式 $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ 证明 $\int_a^b f(x) dx$ (设 $y = a+b-x$ $dx = -dy$)

$$= \int_b^a f(a+b-y) (-dy) \quad \begin{matrix} x=a \text{ 时 } y=b; \\ x=b \text{ 时 } y=a \end{matrix}$$

$$= \int_a^b f(a+b-y) dy$$

$$= \int_a^b f(a+b-x) dx \quad \text{定积分与积分变元无关.}$$

题号

P118

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二 12 求定积分 $\int_0^\pi (e^{\cos x} - e^{-\cos x}) dx$

$$\text{解: } \int_0^\pi (e^{\cos x} - e^{-\cos x}) dx$$

$$= \int_0^\pi [e^{\cos(0+\pi-x)} - e^{-\cos(0+\pi-x)}] dx$$

$$= \int_0^\pi (e^{-\cos x} - e^{\cos x}) dx$$

$$= - \int_0^\pi (e^{\cos x} - e^{-\cos x}) dx$$

$$\Rightarrow \int_0^\pi (e^{\cos x} - e^{-\cos x}) dx = 0$$

翻转公式

(见 P117 11 解答)

三 1 证明题

$$\text{设 } f(x) = \int_1^x \frac{\ln(1+t)}{t} dt, \text{ 证明 } f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2} \ln^2 x$$

$$\text{证明: 设 } F(x) = f(x) + f\left(\frac{1}{x}\right) - \frac{1}{2} \ln^2 x = \int_1^x \frac{\ln(1+t)}{t} dt + \int_1^{\frac{1}{x}} \frac{\ln(1+t)}{t} dt - \frac{1}{2} \ln^2 x$$

$$F'(x) = \frac{\ln(1+x)}{x} + \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) - \frac{\ln x}{x}$$

$$= \frac{\ln(1+x)}{x} + \frac{\ln(1+x) - \ln x}{-x} - \frac{\ln x}{x}$$

$$= 0$$

$$F(x) = C, \text{ 又 } F(1) = 0$$

$$\text{所以 } F(x) = 0, \text{ 即 } f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2} \ln^2 x$$

注, 若一个函数的导数为 0, 则该函数为常数函数

三.2 设 $f(x)$ 在 $[0, 1]$ 上连续,

(1) 证明 $\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$

(2) 求 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

(1) 证明: $\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\sin(0 + \frac{\pi}{2} - x)) dx$ 翻转公式
 $= \int_0^{\frac{\pi}{2}} f(\cos x) dx$

(2) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \sqrt{1 - \sin^2 x}} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sqrt{1 - \cos^2 x}} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \left(\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\cos x + \sin x} \right) dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} dx$

$= \frac{\pi}{4}$

翻转公式见 P117 页 11 题